ON ORIGIN AND STATISTICAL CHARACTERISTICS OF 1/F-NOISE

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Abstract. We suggest some principal ideas on origin, statistical properties and theoretical description of 1/f-noise exactly as they for the first time were expounded in our preprint published in Russian in 1982, and supplement them with short today's comments and selected references, with wish to support improvements of present generally poor ideologic and mathematical base of the 1/f-noise theory.

"Many things are incomprehensible not because our notions are poor but because these things are not in a frame of our notions"
(Koz'ma Prutkov)

PREFACE

As far as we know, the present state of the 1/f-noise problem has no principal differences from that thirty years ago when our preprint had appeared (Preprint NIRFI No.157, published in 1982 by the Scientific-Research Radio-Physics Institute in Nijnii Novgorod in Russian Federation). Of course, this is said not about varieties of objects under experimental investigation but about conceptual level of their theoretical interpretation. Reading of today's scientific literature shows us that the interpreters as before are entangled in prejudices cultivated by more than centennial history of hypotheses, assumptions and approximations in statistical physics and kinetics. We in our preprint just made first attempt to disentangle. Therefore we think that it still may be useful for interested readers, all the more that it affected all our later works (first of all [22, 23, 24, 25]).

Later, we have discovered N. Krylov's book [26] which is devoted to disclosure of the prejudices and gave us principal justification of our own findings. In turn, our own investigations of "molecular Brownian motion" ([27] - [39], [44], [48]) and "electron Brownian motion" ([28], [40] - [43], [44], [48]) in statistical mechanics gave confirmation of Krylov's ideas. As the result, in particular, now we have at our disposal more correct and formally substantiated terminology than thirty years ago. Nevertheless, our translation of the preprint is as much "one-to-one" as possible. Some necessary

corrections and additions can be found in the mentioned references and in comments below designed also as references but with mark ✓ .

Contents

Abstract

It is shown that thermodynamically equilibrium Brownian motion of charge carriers always possesses low-frequency fluctuations in diffusivities of the carriers and thus in power spectrum of "white" noise of the conducting media, with correlation function of the fluctuations decaying by logarithmic law $(\ln t/\tau_0)^{-1}$ while their spectrum is of $1/\omega$ type. Similar fluctuations are peculiar to electric conductivity and current in non-equilibrium state (under external field). It is shown that these fluctuations are not related to some slow processes (and macroscopic relaxation times). Statistical characteristics of the 1/f-noise are completely determined by microscopic parameters of "fast" random motion of the carriers. The 1/f-type spectrum reflects absence of long-living correlations at this random motion.

An exhaustive information about spectrum and magnitude of the 1/f-noise is contained in fourth-order cumulant function of equilibrium current fluctuations.

The suggested theory estimates the 1/f-noise intensity in agreement with experimental data and explains origin of the empiric "Hooge constant".

1. Introduction

1.1. **Experimental data and empiric relations.** Electric charge transport in various media and systems is accompanied by characteristic low-frequency noise, - so called excess noise, or flicker noise, - which is known almost as long as usual white noise, but still has no theoretical explanation.

Surprising peculiarity of flicker noise is that its spectral power density is fast increasing as frequency decreases, approximately by law $\omega^{-\alpha}$, and does not show tendency to saturation down to minimal measurable frequencies $\sim 10^{-6} \div 10^{-7}$ Hz. In most cases the exponent α is close to unit, $\alpha \approx 1$, and then it is said about 1/f-noise.

Intensity of flicker fluctuations of electric current $J(t)$ or voltage in weakly non-equilibrium states (in Ohmic regime) is proportional to squared mean current, therefore flicker noise is usually considered as result of fluctuations of resistance or conductance. Main relationships of electric current 1/f-noise in homogeneous conducting media are reflected by approximate empiric "Hooge formula" (for details see reviews [1-2]):

$$S_J(\omega) \,=\, \overline{J}^2 \,\frac{2\pi\, a}{N\,\omega} \tag{1.1}$$

Here $S_J(\omega)$ is spectral power density of current fluctuations, $\overline{J}$ is mean current, N is number of charge carriers in a sample of media, and a is dimensionless quantity.

Formula (1.1) reflects one more surprising property of 1/f-noise: indifference of shape of its spectrum to system's geometry and sizes, the so-called "zero-dimensionality" of 1/f-noise.

Formula (1.1) usually gives satisfactory description of 1/f-noise in semiconductors, solid and liquid metals, electrolytes [1]. In case of semiconductors the quantity a is almost independent on temperature T and the number N. The latter observation is evidence of statistical independence of contributions from particular charge carriers into full 1/f-noise. Interestingly, in various intrinsic (weakly doped) semiconductors a has nearly same order of magnitude, $a \sim 0.001$ (for the first time this fact was noticed by Hooge [3]). At high enough temperatures, values $a \sim 0.001 \div 0.01$ characterize also metals, though there a is temperature-dependent [1]. For electrolytes also one can find $a \sim 0.001$ ✓[49].

There are three sorts of significant differences of 1/f-noise level from that responding to $a \sim 10^{-3}$. In strongly doped semiconductors the noise is much weaker. There, according to Vandamme and Hooge, $a \approx 10^{-3}\,(\mu/\mu_0)^2$, where μ is mobility of carriers and μ_0 their mobility in pure material. In inhomogeneous systems (among which, seemingly, one should rank also very thin films and wires) the noise may be much greater. In metals at comparatively low temperatures one can observe "temperature 1/f-noise" [1,2,4,5]. It is characterized by as dependence on conductivity's temperature coefficient and sample's thermal contact with surroundings. Perhaps, as was noticed in [5], in metals the noise represents superposition of such "temperature 1/f-noise" and the above mentioned noise which dominates at higher T ($T \gtrsim 150^\circ\,\mathrm{K}$).

In general, the picture of flicker noise in metals looks much more complicated than in semiconductors. In many cases measurements of the exponent α show values visually different from one ($0.8 \lesssim \alpha \lesssim 1.2$), so that formula (1.1) appears too rough.

1.2. **Correlation experiments. "Zero-dimensionality" of 1/f noise.** "Zero - dimensionality" of 1/f-noise expressively manifests itself in so-called correlation experiments: small neighbor regions of same conducting sample produce uncorrelated contributions to its 1/f-noise. This seems striking, since we speak about extremely slow, in microscopic time scale, fluctuations. Indeed, 1/f-type spectrum can be expanded into sum of Lorentzians,

$$\frac{1}{\omega} \rightarrow \frac{2}{\pi} \int_{\Omega_0}^{\infty} \frac{\Omega}{\omega^2 + \Omega^2} = \frac{1}{\omega} \left(\frac{2}{\pi} \arctan \frac{\omega}{\Omega_0} \right) \,,$$

were, as was mentioned, $2\pi/\Omega_0 > 10^6\,$s. And we may assume that there is some real fluctuation mode beyond either Lorentzian, with relaxation time $2\pi/\Omega$. However, it is hard to imagine flutuations' mechanism what would be space-local but at that lead to time scales up to $10^6\,$s (if not greater) ✓[51].

Though, the "temperature 1/f-noise" in metals possesses spatial correlations [4]. Therefore Voss and Clarke concluded in [4] that it is caused by mere thermodynamic fluctuations of temperature "modulating" conductivity. Now, one can state that this conclusion was wrong. The theory of temperature fluctuations, consistent with principles of statistical thermodynamics, does not lead to 1/f-type spectra [1,2,4]. At the same time, there are no doubts that the "temperature 1/f-noise" is closely related to thermal processes.

It is easy to accept the complexity of situation with metals. In metals, electrons are not only charge carriers but also main heat carriers, so that electric and thermal processes are mutually entangled.

1.3. **Thermodynamically equilibrium 1/f-noise and fourth-order cumulant of current fluctuations.** The accumulated experimental data give evidence that 1/f-noise is by its nature thermodynamically equilibrium [1,2]. This principally important circumstance means that unknown processes, what are responsible for 1/f-noise, take place first of all in equilibrium state, although do not manifest themselves in correlation function and spectrum of electric current or e.m.f.

Nevertheless, in some things these processes must be reflected even in equilibrium case. It is not hard to guess that they must lead to flicker fluctuations of intensity of equilibrium white noise, $S(t)$. This is prompted already by the Nyquist formula: $S = 2Tg$, if one interprets 1/f-noise as consequence of fluctuations of conductance, $g(t)$. Voss and Clarke measured fluctuations of white noise power in thin metal film and really found that they have 1/f-type spectrum [4]. This phenomenon also can be termed 1/f-noise (equilibrium now just in literal sense). In out-of-equilibrium system under external field it transforms into current (and voltage) fluctuations which will be termed below current 1/f-noise.

It should be underlined that the fluctuating power $S(t)$, in contrast to $J(t)$, is especially phenomenological characteristics of noise. While $J(t)$ always can be written as a function of microscopic dynamical variables of a system, $S(t)$ can not be expressed by dynamical language neither through these variables nor through fluctuations of thermodynamical quantities: temperature, chemical potential, etc., - characterizing

system's quasi-equilibrium states. The matter is that the power (power spectral density) of white noise represents kinetic quantity. Its definition should involve statistical averaging over ensemble and, besides, integration (averaging) over time. But in such way one can rigorously introduce only mean value [1]

$$\langle S(t) \rangle_0 \equiv S_0 = \int_{-\infty}^{\infty} \langle J(t)\, J(0) \rangle_0 \, dt \,,$$

while fluctuations of $S(t)$ have no strictly definite dynamical sense. The said concerns also fluctuations of conductance $g(t)$ and other kinetic quantities.

Then, how we can rigorously describe fluctuations of the power and conductance? It is very simple issue. Since $\langle S(t) \rangle_0$ is connected to quadratic forms, in respect to current or voltage, fluctuations of $S(t)$ are corresponded by current's statistical moments of fourth order (and higher orders).

Hence, it is necessary to consider fourth moment of current's fluctuations:

$$\begin{aligned} \langle J(t_1)\, J(t_2)\, J(t_3)\, J(t_4) \rangle_0 \,=\, \langle J(t_1)\, J(t_2) \rangle_0 \langle J(t_3)\, J(t_4) \rangle_0 \,+ \\ + \langle J(t_1)\, J(t_3) \rangle_0 \langle J(t_2)\, J(t_4) \rangle_0 + \langle J(t_1)\, J(t_4) \rangle_0 \langle J(t_2)\, J(t_3) \rangle_0 \,+ \\ + \langle J(t_1)\,,\, J(t_2)\,,\, J(t_3)\,,\, J(t_4) \rangle_0 \end{aligned} \tag{1.2}$$

For it always there is a rigorous dynamical expression. The brackets with commas inside on the right in (1.2) mean fourth-order cumulant of current. Equality in (1.2) is known general cumulant expansion of fourth moment at $\langle J(t) \rangle_0 = 0$.

We have to underline that information about the power fluctuations and equilibrium 1/f-noise is hidden in the last term of (1.2), i.e. fourth cumulant, which characterizes non-Gaussianity of current fluctuations. The equilibrium white noise correlation function $\langle J(t)\, J(t_2) \rangle_0$ is microscopically fast decaying under increase of $|t_1 - t_2$. Therefore low-frequency processes are reflected by only last term of (1.2).

From here an important consequence does follow, that in statistical description of 1/f noise it is very necessarily to take into account non-Gaussianity of current fluctuations (even if it is very weak in one or another case) [2]. Of course, not current only, but also any other physical random process always is more or less non-Gaussian. Here we meet the situation when this statement is of principal importance.

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[1] Subscript "0" at angle brackets indicates that averaging is made over equilibrium ensemble.

[2] Notice that, for instance, Gaussian noise with randomly varying intensity becomes non-Gaussian random process. For more about analysis of noise non-Gaussianity see also: M.Nelkin and A.M.S.Tremblay, J. Stat. Phys. **25** 253 (1981).

1.4. **Fluctuations of mobility of carriers.** A number of authors, basing on analysis of experiments, have come to conclusion that primary source of 1/f-noise is fluctuations of mobilities of carriers [1,2,6]. The Hooge formula appears if one assumes that particular mobilities fluctuate independently one on another with spectrum

$$S_\mu(\omega) = \frac{2\pi\, a}{|\omega|}\, \mu^2 \;, \tag{1.3}$$

where μ is mean value of mobility, $a \approx 10^{-3}$. This model allows also empirical description of 1/f-noise in non-uniform structures: in various contacts, $p - n-$junctions and other semiconductor devices (and, seemingly, in electron emission), moreover, even in non-Ohmic regime [1]. There, as one can think, 1/f-noise arises because of fluctuations of carriers' flow to a structural change in turn caused by mobility fluctuations.

In applications to semiconductors, the hypothesis of mobility fluctuations in many cases meets competitive hypothesis about fluctuations in number of carriers due to slow tunnel transitions to near-surface states and back [1,2,7]. In this model it is easy to obtain 1/f-type spectrum as the sum of Lorentzians (thogh saturating at very low frequencies), but it is hard to obtain numeric estimates. This model contradicts to observed bulk character of 1/f-noise. Besides, special experiments with hot carriers also give evidences in favor of hypothesis of mobility fluctuations [6].

Then, natural question about physical origin of the mobility fluctuations does arise. If ascribing their ground to some slow processes in crystal lattice (e.g. fluctuations in number of phonons ✓[52], as proposed in [1]), one would have to expect mobility fluctuations of different carriers to be correlated one with another. But experiments say about the opposite. If, however, we state absence of inter-carrier correlations, then the known "zero-dimensionality" paradox arises: why it is possible to observe flicker correlations much longer than mean time of residence of charge carriers in a small sample? (It seems as if carrier leaves a sample but nevertheless its correlations, associated with 1/f noise, stay there.) Up to now in the literature none mechanism of the mobility fluctuations was suggested. Evidently, the mentioned paradox strongly complicates the problem ✓[53].

1.5. **Principal statements of present work. 1/f-noise as result of absence of long-living correlations.** In spite of many-year experimental investigations and many attempts of theoretical explanation, 1/f-noise still remains mysterious [1,2]. It is all the more strange in view of that in other respects the systems under question do not display mysterious effects what could be associated with 1/f-noise.

Various investigators' efforts are invariably directed either to search of some new "slow" physical mechanisms, which would lead to wide set of very large relaxation times (or correlation times, "life-times", etc.) and flicker spectrum of conductance fluctuations, or to "approbation" in this sense of already known mechanisms, such as, for example, fluctuations of temperature and carriers concentration (see e.g. [2,4,8-11]). This way does not lead to success.

In the present work, exactly opposite approach to the 1/f-noise problem is suggested and substantiated, allowing to explain this physical phenomenon. It is shown that 1/f-noise can be connected just to absence of macroscopically large relaxation times and caused not by specifically slow processes but by Brownian motion of charge carriers in itself, i.e. the same "fast" microscopic processes only which produce diffusion and white noise.

We show that Brownian motion (diffusion) of carriers always is accompanied by flicker fluctuations of diffusivity and mobility (white noise power and conductance), which naturally and inevitably arise in the course of diffusion irrespective to its concrete microscopic mechanism, without any slow perturbations (as e.g. random changes of thermodynamical conditions of diffusion). The theory to be expounded below not only yields 1/f-type spectrum but also ensures correct estimate of level of 1/f-noise and explains magnitude of the empirical "Hooge constant", $a \sim 10^{-3}$.

The key, principal, idea of our approach is that 1/f-type spectrum is not consequence of real long-living correlations, but, oppositely, results from absence of such correlations, indifference of system to random deviations of "rate" of carrier's diffusion from its mean regime. Such deviation is not suppressed by backmoving forces, since the only its consequence is mere spatial displacement of carrier, that is system's transition into a state which is identical in thermodynamical sense to initial state.

Just in such way, likely, one can explain the observed (see above) mobility fluctuations. The formulated idea brings also solution of the "zero-dimensionality paradox". Since in reality carrier motion is free of any long-living correlations (has no long memory about the past), there is no need to speak about breaking of such correlations [3] under replacements of one carriers in small "noising" sample by others (by similar

[3] It is clear that, very generally, indifference of a system to some spontaneous deviations and their accumulation with time (Brownian displacement of a carrier, in our case) can be formally treated both as absence of correlations and presence of infinitely long correlations, although physically the first of these statements is true. Formal (but not essential!) analogy is given by a superconductor with non-decaying currents.

reasons, the carriers' generation-recombination processes do not destroy the 1/f-noise) ✓[55].

2. Brownian motion and equilibrium 1/f-noise. Fluctuations of white noise power

2.1. **Phenomenological description of current fluctuations.** Let us consider a simple physical situation as follows. Taking a sample of conducting medium, let us short-circuit it by closing it into ring. We are interested in thermodynamically equilibrium fluctuations of electric current $J(t)$ in this closed circuit. We shall assume that the medium is statistically homogeneous and that charge carriers move statistically independently one on another (which usually agrees with real situation in semiconductors). Then it is sufficient to consider random walk of one separate carrier along the ring-like circuit.

Introduce designation $v(t)$ for random velocity of carrier in the ring's direction (thus $v(t)$ will be scalar), and

$$r(t) = \int_0^t v(t')\, dt' \tag{2.1}$$

for its path during time t. Notice that according to (2.1) $r(t)$ counts (with plus or minus sign) any complete rotation around the circuit. In the same sense we shall treat displacement $r(t)$ under charge transfer by not free but localized carriers (hopping conductivity) when $v(t)$ is less natural characteristics of the motion. By this $r(t)$'s definition, conveniently, in a stationary state $r(t)$ is random process with uniform increments (since $v(t)$ is stationary process). In other words, $r(t)$ represents unbounded diffusion, although the system under consideration is essentially bounded. In the very beginning we would like underline that our results will be quite insensitive to sizes of the system (length of the ring).

Of course, $v(t)$ always is more or less non-Gaussian random process. The simplest usually exploited model of diffusion ignores this obvious physical fact and assumes $v(t)$ and $r(t)$ Gaussian. This can be justified when considering carriers concentration fluctuations (just they are taken in mind in literature when speaking about "diffusive noise", "diffusive mechanisms", etc.) but not in analysis of electric noise caused by random wandering of carriers (in our system total number of "noising" carriers does not change during diffusion). As already was pointed out, in Gaussian model intensity of noise is a priori constant.

Full statistical information about fluctuations of $v(t),\ r(t),\ J(t)$ in stationary equilibrium state is contained in characteristic functions (CF)

$$\Theta_t(ik) \,\equiv\, \langle\, e^{ik\,r(t)}\,\rangle_0 \;, \tag{2.2}$$

$$\Theta_t(iu) \,\equiv\, \langle\, e^{iu\,Q(t)}\,\rangle_0 \;, \quad Q(t) \,\equiv\, \int_0^t J(t')\,dt' \tag{2.3}$$

Here k , u are arbitrary probe parameters. It should be underlined that in principle one always can concretize rigorous microscopic expressions for these mathematical objects, therefore use of CF by itself does not presume any approximations. However, in practice one can not do without some assumptions leading to simple enough stochastic model.

In Gaussian model, as is well known,

$$\Theta_t(ik) \,=\, e^{-\,D\,k^2 t} \;, \quad \Theta_t(iu) \,=\, e^{-\frac{1}{2}\,S\,u^2 t} \;, \tag{2.4}$$

where D is diffusivity (diffusion coefficient) and S is spectral power density of equilibrium current noise at zero frequency. Here it is assumed that $t \gg \tau_\mu$, where τ_μ is correlation time of equilibrium noise. More accurate form of (2.4) can be presented with the help of functions

$$\Delta_t(ik) \,\equiv\, \frac{1}{t}\,\ln\langle\, e^{ik\,r(t)}\,\rangle_0 \,=\, \frac{1}{t}\,\ln\,\Theta_t(ik) \;, \tag{2.5}$$
$$\Delta_\infty(ik) \,\equiv\, \lim_{t\to\infty}\frac{1}{t}\,\ln\langle\, e^{ik\,r(t)}\,\rangle_0 \;,$$

and similarly for current. Then in Gaussian model

$$\Delta_\infty(ik) \,=\, -D\,k^2 \tag{2.6}$$

Probability density distribution of the path $r(t)$,

$$W_t(r) \,=\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikr}\,\Theta_t(ik)\,dk \;, \tag{2.7}$$

in this model at $t \gg \tau_\mu$ is

$$W_t(r) \,=\, (4\pi Dt)^{-1/2}\,\exp\left(-\frac{r^2}{4Dt}\right) \tag{2.8}$$

Now let us consider on-Gaussian model which takes into account fluctuations of intensity of current noise ✓[56]. The concept of such fluctuations is meaningful only when they are very slow from viewpoint of the scale τ_μ . Therefore their phenomenological description requires to use a model treating $v(t)$ like Gaussian white noise with random "modulation' of intensity $2D(t)$ (after whuch $v(t)$ becomes non-Gaussian).

In such model by definition

$$\left\langle \exp\left\{ \int_0^t ik(t')\, v(t')\, dt' \right\} \right\rangle_0 =$$
$$= \left\langle \exp\left\{ -\int_0^t D(t')\, k^2(t')\, dt' \right\} \right\rangle_0' = \qquad (2.9)$$
$$= \exp\left\{ -D \int_0^t k^2(t')\, dt' + \frac{1}{2}\int_0^t \int_0^t K_D(t'-t'')\, k^2(t')\, k^2(t'')\, dt' dt'' + \dots \right\} ,$$

where the first equality corresponds to averaging over white noise at fixed function $D(t)$, the brackets $\langle \dots \rangle_0'$ denote averaging over $D(t)$'s fluctuations, $D = \langle D(t) \rangle_0'$,

$$K_D(t'-t'') \,=\, \langle\, D(t')\, D(t''))\, \rangle_0' \,-\, D^2 \,\equiv\, \langle\, D(t')\,,\, D(t''))\, \rangle_0'$$

is correlation function of fluctuations of diffusivity $D(t)$, the dots replace contributions to CF from higher-order cumulants of fluctuations $D(t)$, and $k(t)$ is arbitrary probe function.

On the other hand, there is general exact expansion of logarithm of the CF (2.9) into series over cumulants of the velocity [4]:

$$\left\langle \exp\left\{ \int_0^t ik(t')\, v(t')\, dt' \right\} \right\rangle_0 = \qquad (2.10)$$
$$= \exp\left\{ \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_0^t \langle\, v(t_1)\,,\, \dots\,,\, v(t_n)\, \rangle_0\; k(t_1)\dots k(t_n)\, dt_1 \dots dt_n \right\}$$

In fact, this is definition of the cumulants [13,14]. In equilibrium $\langle v(t) \rangle_0 = 0$. Equating (2.9) to (2.10), in view of $k(t)$'s arbitraryness, one can obtain for fourth-order velocity cumulant

$$\langle\, v(t_1)\,,\, v(t_2)\,,\, v(t_3)\,,\, v(t_4)\, \rangle_0 \,=\, 4\,\delta(t_1-t_2)\,\delta(t_3-t_4)\, K_D(t_1-t_3) \,+ \qquad (2.11)$$
$$+\, 4\,\delta(t_1-t_3)\,\delta(t_2-t_4)\, K_D(t_1-t_4) \,+\, 4\,\delta(t_1-t_4)\,\delta(t_2-t_3)\, K_D(t_1-t_2)$$

From here we find (with taking into account that $\int_0^\infty \delta(t)\, dt \,=\, 1/2$)

$$\int_0^\infty \int_0^\infty \langle\, v(t)\,,\, v(t')\,,\, v(t'')\,,\, v(0)\, \rangle_0\; dt'\, dt'' \,=\, 2K_D(t) \qquad (2.12)$$

Clearly, in analogous model of fluctuations of current $J(t)$ for its fourth-order cumulant appearing in (1.2) we must obtain

$$\int_0^\infty \int_0^\infty \langle\, J(t)\,,\, J(t')\,,\, J(t'')\,,\, J(0)\, \rangle_0\; dt'\, dt'' \,=\, \frac{1}{2}\, K_S(t)\ , \qquad (2.13)$$

[4] Angle bracket with comma-separated factors inside (“Malakhov’s cumulant bracket”) means joint cumulant of these factors.

where $K_S(t)$ is correlation function of fluctuating power $S(t)$ of equilibrium white noise.

2.2. **Non-Gaussian random walk. Characteristic function of displacement.** Let us consider CF (2.2). Its logarithm (2.5) for brevity also will be termed CF. Expanding CF (2.5) into series over ik we have

$$\Delta_t(ik) = \sum_{m=1}^{\infty} \frac{(ik)^{2m}}{2m\,!} D_{2m}(t) \tag{2.14}$$

We took into account that, because of invariance of laws of microscopic motion in respect to time reversal, equilibrium Brownian motion is spatially symmetric. Therefore in (2.14) only even degrees do appear. From the probability theory it is known that full number of nonzero terms of the series (2.14) is always infinite. The only exclusion is "Gaussian" case when $D_n(t) = 0$ for all $n \geq 3$.

According to (2.1), (2.5), (2.10),

$$D_n(t) = \frac{1}{t} \int_0^{\infty} \langle\, v(t_1)\,,\, \dots\,,\, v(t_n)\,\rangle_0 \; dt_1 \dots dt_n \tag{2.15}$$

If observation time $t \gg \tau_\mu$ then $D_2(t) = 2D$, where D is diffusivity. Assume that higher correlators (cumulants) of velocity are fast enough decaying at $|t_i - t_j| \rightarrow \infty$. In such case in (2.14)-(2.15) limits $D_n \equiv \lim_{t\rightarrow\infty} D_n(t)$ do exist. The quantities D_n give values of n-order poly-spectra at zero frequencies, they can be termed also n-order "non-Gaussian diffusion coefficients". From (2.5), (2.14) we have

$$\Delta_\infty(ik) = \sum_{m=1}^{\infty} \frac{(ik)^{2m}}{2m\,!} D_{2m} = -D\,k^2 + D_4 \frac{k^4}{24} - \dots \tag{2.16}$$

If, however, higher-order velocity correlators by some reason are slow decaying under separation of time arguments, then the diffusion coefficients D_n ($n \geq 4$) may become infinite. It means that CF (2.16) is non-analytical function of ik. In such the case it is convenient to use integral representation of CF [12]:

$$\Delta_\infty(ik) = \int_{-\infty}^{\infty} (\cos\, kr - 1)\, \frac{2D}{r^2}\, G(r)\, dr \tag{2.17}$$

The multiplier $2D$ here is extracted by dimensionality reasonings. From (2.14)-(2.15) it follows that $G(-r) = G(r)$ and

$$\int_{-\infty}^{\infty} G(r)\, dr = 1 \tag{2.18}$$

In essence, (2.17) is a kind of Fourier transform (taking into account that $\Delta\infty(0) = 0$).

In the probability theory the limit transition $\tau_\mu/t \to 0$, analogous to the above concerned one, is performed in another, more formal, fashion. There t stays finite and at that τ_μ is turned to zero. The result is some "Brownian" process with infinitely divisible increments, i.e. with independent increments [12] ✓[57]. The integral representation like (2.17) is termed Levy-Khinchin representation. A fundamental theorem of the probability theory states that the kernel $G(r)$ always is non-negative, $G(r) \geq 0$, which just means the infinite divisibility. In other respects this function is arbitrary (if only the integral in (2.17) is converging).

The Gaussian diffusion, when $D_n = 0$ for $n \geq 3$, is single peculiar, degenerated, case corresponding to kernel $G(r) = \delta(r)$.

Under "physical" limit transition, when t grows while τ_μ is fixed, the limit of CF (2.10), generally speaking, may not correspond to strictly infinitely divisible distribution, and $G(r)$ can be not strictly non-negative. However, in various physical problems the increments of $r(t)$, $Q(t)$, etc., at large t practically possess asymptotic property of infinite divisibility (even in presence of slow decaying, non-integrable, correlations; see example in [15]).

CF (2.16) contains complete information about large-scale characteristics of the random walk considered in rough macroscopic time scale. Corresponding approximate expression for probability distribution of $r(t)$ follows from (2.7) after replacement $\Theta_t(ik) \to \exp(t\Delta_\infty(ik))$. At $t \to \infty$, $k \to 0$, a dominating role is played by first term of the expansion (2.16), that is diffusion is asymptotically Gaussian.

2.3. **Scale invariance of real Brownian motion ($r^2 \propto t$).** The Brownian motion under consideration represents a physical process realizing in a thermodynamical system and in equilibrium state. Like other thermodynamical phenomena, this process must not depend (on scales much greater than characteristic microscopic scales) on detail structure of microscopic interactions. Consequently, it should possess some spatial-temporal scale invariance. A form of this scale invariance is evidently indicated by dimensionality of diffusion coefficient D, the macro-parameter determining large-scale properties of Brownian motion (and characteristic law of diffusion, $\langle r^2(t)\rangle_0 = 2Dt$).

Another parameter, what would compete with D in this sense, could appear only due to some slow physical processes having significant influence on statistics of diffusion.

We assume that there are no such slow processes in our system. Then the only additional parameters determining (together with D) complete set of statistical characteristics of diffusion $\{D_n(t)\}$ are microscopic quantities describing "fast" interactions

of carrier with medium. This means that the scale invariance claimed by the "average" law of diffusion, $\langle r^2(t)\rangle_0 = 2Dt$, governs the whole statistical picture of Brownian motion. In other words, it must look self-similar when spatial scale changes by λ times while temporal scale by λ^2 times ($r^2 \propto t$).

Mathematically this statement reads, as one cam see from (2.5), as follows:

$$\lambda^2 \, \Delta_{\lambda^2 t} \left(\frac{ik}{\lambda} \right) \, = \, \Delta_t(ik) \ , \tag{2.19}$$

at sufficiently large t and small $|k|$, and

$$\lambda^2 \, \Delta_\infty \left(\frac{ik}{\lambda} \right) \, = \, \Delta_\infty(ik) \tag{2.20}$$

Below we shall show that such scale invariance of Brownian motion (at large scales) realizes through flicker fluctuations of diffusivity (rate of the motion) with 1/f-type spectrum.

2.4. **1/f-noise as natural attribute of diffusion. Spontaneous diffusivity fluctuations. Logarithmically decaying correlations.** First, consider relation (2.20). Substitution of (2.17) to (2.20), after change of variables in the integral we obtain

$$\lambda \, G(\lambda r) \, = \, G(r)$$

This functional equation has two solutions:

$$G(r) \, = \, \delta(r) \ , \quad G(r) \, = \, \frac{A}{|r|} \ , \tag{2.21}$$

where $A = \text{const}$. The first possibility leads to CF (2.6), i.e. to ideally Gaussian diffusion which has no place in nature. Therefore let us consider the second possibility.

Choice of the second of expressions (2.21) result in divergency of integral in (2.17) at $r \to 0$. This means that the scale invariance can not be perfect, that is it must be violated at small (microscopic) scales, which is obvious from physical viewpoint. Consequently, we have to cut-off the integrand, for instance, by setting

$$G(r) \, = \, \frac{A}{|r| + r_0} \tag{2.22}$$

(below it will be seen that details of cut-off procedure are rather insignificant). Thus we introduce characteristic spatial micro-scale r_0. It can not be smaller than mean free path (in case of quasi-free carriers) or mean step under hopping conductivity ✓[58].

Inserting (2.22) to (2.17), we obtain, at $k^2 r_0^2 \ll 1$ (which corresponds to much larger scales than r_0),

$$\Delta_\infty(ik) \approx -Dk^2 A \ln \frac{1}{r_0^2 k^2} \tag{2.23}$$

✓[59] . Non-analyticity of this function manifests presence of non-integrable long-living higher-order correlations of $v(t)$. At that, however, it is unpleasant that function (2.22) does not satisfy the necessary condition (2.18), and therefore expression (2.23) does not identify quadratic term $\propto (ik)^2$. This means that the invariance must be destroyed at large scales too, as far as observations of the diffusion process take a finite time.

Hence, we have to go back to CF (2.5) for finite time and analyse it with the help of relation (2.19). With this purpose, let us use analogue of the representation (2.22) as follows:

$$\Delta_t(ik) = \int_{-\infty}^{\infty} (\cos kr - 1) \frac{2D}{r^2} G(r,t)\, dr \ , \tag{2.24}$$

$$\int_{-\infty}^{\infty} G(r,t)\, dr = 1 \tag{2.25}$$

At $t \to \infty$ function $G(r,t)$ should turn into (2.22) (but now with coefficient A depending on t because of condition (2.25). The condition (2.25) says that at $k \to 0$ (and $t \gg \tau_\mu$) CF (2.25) tends to expression $-Dk^2$, i.e. is asymptotically Gaussian. One can verify that these reasonings together with (2.19) and (2.24) imply the following form of the kernel in representation (2.24):

$$G(r,t) = \frac{A(t)}{|r| + r_0} F\left(\frac{r^2}{4D' t}\right) \ , \tag{2.26}$$

where D' is a constant with same dimensionality as D has, $A(t)$ is determined by the normalization condition (2.25), function $F(z)$ satisfies requirements

$$F(0) = 1 \ , \quad F(z) \to 0 \ \texttt{at} \ z \to \infty \ , \quad \int_0^\infty F(z)\, dz = 1 \tag{2.27}$$

(they always can be satisfied due to presence of free parameters $A(t)$, D'). With use of (2.27) we find from (2.25) that, regardless of concrete form of $F(z)$,

$$A(t) = \left\{ \int_{-\infty}^{\infty} F\left(\frac{r^2}{4D' t}\right) \frac{dr}{|r| + r_0} \right\}^{-1} \approx \left(\ln \frac{t}{\tau_0}\right)^{-1} \tag{2.28}$$

at $t \gg \tau_0$. Here a microscopic time scale has appeared:

$$\tau_0 = \frac{r_0^2}{2D'} \tag{2.29}$$

Next, insert (2.26), (2.28) to (2.24) and consider first terms of expansion (2.24):

$$\Delta_t(ik) \,=\, -Dk^2 \,+\, \frac{1}{3}\,DD'\,t\,A(t) \,-\, \dots \;,$$
$$D_4(t) \,=\, 8\,DD'\,tA(t) \tag{2.30}$$

The second term of series (2.30) contains information about diffusivity fluctuations. Indeed, phenomenological relations (2.11), (2.12) and definition (2.15) imply

$$\frac{d^2}{dt^2}\,tD_4(t) \,=\, 24\,K_D(t)\;, \tag{2.31}$$

and then comparison of (2.30) and (2.31) yields (for $t \gg \tau_0$)

$$K_D(t) \,=\, \frac{1}{3}\,DD'\,\frac{d^2}{dt^2}\,t^2\left(\ln\,\frac{t}{\tau_0}\right)^{-1} \approx\, \frac{2}{3}\,DD'\left(\ln\,\frac{t}{\tau_0}\right)^{-1} \tag{2.32}$$

Thus, correlation function of diffusivity fluctuations decays by logarithmic law. It is obvious, already from dimensionality considerations, that corresponding spectrum is of 1/f-type.

We see that the flicker fluctuations appear as generic property of real Brownian motion (which inevitably becomes scale-invariant when all microscopic scales "are left behind") ✓[61].

Interestingly, these fluctuations rather weakly tell on shape of the displacement's probability distribution (2.7). As a not complicated analysis of (2.24), (2.26) does show, at $t \gg \tau_0$ and $k^2 r_0^2 \ll 1$ CF (2.24) has approximately universal form

$$\Delta_t(ik) \,=\, \frac{Dk^2}{\ln\,\frac{t}{\tau_0}}\,\ln\left(r_0^2 k^2 \,+\, c\,\frac{\tau_0}{t}\right)\;, \tag{2.33}$$

where c is a quantity of order of unit. CF (2.33) rather weakly differs from ideally Gaussian one, (2.6), although coefficients of expansion of (2.33) into series (2.14) unboundedly grow with time. Correspondingly, difference between distribution (2.7), resulting from (2.33), and Gaussian "bell" (2.8) is almost unnoticeable. In other words, flicker fluctuations $D(t)$ do not destroy usual picture of diffusion. If we "let out" an ensemble of Brownian particles to walk from some point, their distribution law (evolution of their concentration) is almost Gaussian bell (2.8) ✓[62]. Moreover, it can be shown that fluctuations $D(t)$ practically have no influence on (equilibrium) concentration fluctuations, so that the latter can be well described in the frameworks of usual "ideally Gaussian" model of diffusion.

2.5. **Spectrum of diffusivity fluctuations. Quantitative estimates. Origin of the "Hooge constant".** Consider spectral power density of (relative) fluctuations of diffusivity,

$$S_{\delta D}(\omega) \,\equiv\, \frac{S_D(\omega)}{D^2} \,\equiv\, \frac{2}{D^2}\int_0^\infty K_D(t)\,\cos\,\omega t\,dt \tag{2.34}$$

We omit calculation of the Fourier integral. The result, in the region of our interest ($\omega\tau_0 \ll 1$), with good accuracy is equal to

$$S_{\delta D}(\omega) \,=\, \frac{2D'}{3D}\,\frac{\pi}{|\omega|\,(\,\ln\,|\omega|\tau_0\,)^2} \tag{2.35}$$

Let us compare this expression with the empirical formula (1.3). Slow flicker fluctuations $D(t)$ and fluctuations of mobility, $\mu(t)$, should be connected, as it follows from simple physical reasonings, by the Einstein relation $D(t) = T\mu(t)$ (in Sec.3 we shall rigorously prove it for steady non-equilibrium state). Consequently, $S_{\delta D}(\omega) = S_{\delta\mu}(\omega)$, and we can rewrite (2.35) as

$$S_{\delta\mu}(\omega) \,=\, \frac{2\pi\,a(\omega)}{|\omega|}\;\;, \tag{2.36}$$

$$a(\omega) \,\equiv\, \frac{D'}{3D}\,(\,\ln\,|\omega|\tau_0\,)^{-2} \,=\, \frac{r_0^2}{6D\tau_0}\,(\,\ln\,|\omega|\tau_0\,)^{-2}$$

Spectrum of relative fluctuations of power, $S(t)$, of summary white noise produced by N statistically independent carriers results from (2.35) after division by N:

$$S_{\delta S}(\omega) \,\equiv\, S_{\delta D}(\omega)\,\frac{1}{N} \tag{2.37}$$

The parameters r_0 and τ_0 are micro-scales representing lower bounds of the scale invariance of Brownian motion. Physically, it seems obvious that in homogeneous medium $2D\tau_0 \gtrsim r_0^2$, i.e. $D' \lesssim D$. Indeed, if already after single typical "free path" (or "hop"), with length λ_0, self-correlation of direction of motion vanishes, then r_0 must turn to λ_0, and therefore quantity $2D\tau_0$ can not be essentially smaller than $\lambda_0^2 \approx r_0^2$ ✓[63].

If temporal invariance takes shape starting just from minimal accessible scale $\sim r_0^2/2D$, then $2D\tau_0 \approx r_0^2$ and $D' \approx D$. In this simplest case $\tau_0 \approx \tau_\mu$ (with τ_μ denoting typical free path time, or time between hops from one localized state to another). At that, "width" of function $F(r^2/4D't)$ in (2.26) coincides, in view of (2.27), with width of Gaussian bell (2.8). In this remarkable characteristic case, diffusion is invariant to maximal extent, since it is described by only two parameters, D and τ_0.

Confining ourselves by this situation [5], let us set $2D\tau_0 = r_0^2$ ($D' = D$) and estimate the "Hooge constant" $a(\omega)$ (compare (2.36) with (1.3)).

In semiconductors, typical free path time $\tau_\mu \sim 10^{-12}\,$s. Taking $\tau_0 \sim 10^{-12}\,$s, at frequency $\omega/2\pi = 1\,$Hz we obtain $a = (1/3)(\ln \omega\tau_0)^{-2} \approx 5\cdot 10^{-4}$, and $a \approx 1.2\cdot 10^{-3}$ at frequency $\omega/2\pi = 10^4\,$Hz, in good agrrement with typical experimental values $\sim 10^{-3}$ (see Introduction). Thus our theory gives correct estimate of the Hooge constant and, hence, 1/f-noise level in many real situations [1].

We see that the "Hooge constant", $a \approx (1/3)(\ln \omega\tau_0)^{-2}$, is determined by microscopically small time τ_0 and total duration $\sim 2\pi/\omega$ of 1/f-noise observations only. Frequancy dependence of $a(\omega)$ is very weak. But it ensures integrability of spectrum (2.35)-(2.36) at low frequency (as it should be for stationary fluctuations) ✓[64].

If $2D\tau_0 \gg r_0^2$ then, according to (2.36), 1/f-noise level essentially decreases. This statement agrees with observed lowering of noise in strongly doped semiconductors [1]. Indeed, there r_0 is determined by carriers' scattering by impurities. However, the scale τ_0 must be determined by much more slow interaction with phonons. Therefore $2D\tau_0 \gg r_0^2$. Let us assume that $2D\tau_0 \approx \lambda_0^2$ with λ_0 being mean free path under interaction with phonons only (i.e. in pure material). Then in (2.36) $r_0^2/2D\tau_0 \approx r_0^2/\lambda_0^2$, which can be written also as $r_0^2/2D\tau_0 \approx (\mu/\mu_0)^2$, in agreement with the empirical Vandamme-Hooge formula (see Introduction) ✓[65].

2.6. **Microscopic origin of flicker fluctuations.** We demonstrated that if Brownian motion has no inner macroscopic scales and therefore possesses natural scale invariance, $r^2 \propto t$, then it is accompanied by flicker fluctuations of its intensity ("rate" of diffusion). Already the very premise implies the statement from Introduction that related long-living correlations are manifestation of absence of long-living memory in random walks of carriers, i.e. these correlations are imaginary (nevertheless, one has to describe them statistically like real correlations) ✓[66]. Both their behavior reflected in (2.32) and logarithmic cut-off of spectrum (2.35)-(2.37) at very low frequencies are determined by the microscale τ_0 only. In essence, the words about "imaginarity" of flicker fluctuations and absence of macroscopic scales are equivalent.

The "microscopic" interpretation of obtained results enforces to refine some conventional notions. The usual view at fluctuations of mobility and conductance is that these kinetic characteristics always, at any time moment, have exactly definite instant

[5] Expectedly, it takes place, in particular, under domination of one type of inelastic interaction of carriers with thermostat, when all correlations of $v(t)$ are fast decaying during time $\sim r_0^2/2D$.

values which, however, undergo some slow random perturbations. Then one should find mechanisms of the perturbations, though such attempts constantly are unsuccessful. But there exists quite different point of view: kinetic characteristics have no certain "instant" values, and just this physical fact explains 1/f-noise.

Really, it is not hard to understand that interaction of Brownian particle (carrier) with thermostat not only impels it to diffusive motion but simultaneously causes deviations of its random motion from some average regime. The latter can be definitely and quickly measured if observing a large ensemble of carriers, as is usually done in experiments. However, a separate carrier "knows nothing" about properties of the ensemble and is not obligatory at all to display walk with certain diffusivity and mobility. The latter concepts have a meaning only in respect to a large enough set of realizations of random motion, while their application to individual random (but dynamical) trajectory of carrier is senseless. A single separate carrier merely has no instant diffusivity and mobility, so that one can speak about fluctuations of these kinetic quantities in suitable phenomenological language only. Both D , μ and correlation functions $K_D(t)$, $K_\mu(t)$, etc., are characteristics of ensemble of trajectories.

Of course, if a thing does not exist (as clear dynamical characteristics of motion) then it can not fluctuate. Therefore we have to complicate the picture. The diffusion's property what is empirically perceived as flicker fluctuations $D(t)$ (or $\mu(t)$) is mere consequence of temporal uniformity of Brownian motion, thermodynamical indifference of the system to "rate" of diffusion. Wherever a carrier has occurred at given time moment, it each time "starts from beginning", and its past is of no importance. Therefore any deviations from ensemble-average regime of motion do not induce a compensation and accumulate in time [6] . These deviations, or fluctuations of "degree of randomness" of motion, cause neither dynamical nor thermodynamical reaction of the system ✓[71]. At the same time, they stay in frames of characteristic diffusive law $r^2 \propto t$. The result, as we have seen, is 1/f-noise.

All that picture, which is slipping away from conventional kinetic's ideology, in principle could be described in formally rigorous and complete enough way in terms of the fourth-order correlators (cumulants) (1.2), (2.12), (2.13), if we were able to analyze them by methods of statistical mechanics. However, such the task is extremely complicated, even in case of quadratic correlators.

[6] Notice that in [8] an attempt was made to introduce to a model of diffusion similar non-decaying ("residual") correlations, but they were addressed to fluctuation in concentration of diffusing quantity. In our model, long correlations automatically appear in diffusivity fluctuations $D(t)$.

In essence, all the aforesaid concern any kinetic quantities, not "electric" ones only. Thus, likely, we can say that in any system, where time-uniform transport of some extensive physical quantity takes place, there are flicker fluctuations of spectral power of random flows (Langevin forces) and, in non-equilibrium state, also similar fluctuations of irreversible flows. For example, the heat transfer must be accompanied by flicker fluctuations of thermal conductivity and, under temperature gradient, heat flow. Another example is given by observed fluctuations (with 1/f spectrum) of energy losses in quartz resonators (see e.g. [9]) ✓[72].

2.7. **General description of equilibrium 1/f-noise.** Now, let us take arbitrary two-terminal dissipative electric conductor whose leads are shortly connected. Consider equilibrium diffusion of charge $Q(t)$ (see (2.3)) around this closed circuit. Here, also the usual diffusion law is satisfied on average, $\langle Q^2(t)\rangle_0 = St$. If transport of charge is determined by microscopically scaled processes only, the "charge Brownian motion" $Q(t)$ again must be scale-invariant. Obviously, now we can immediately write an expression for spectrum, $S_{\delta S}(\omega)$, of relative fluctuations of the white noise power. It is sufficient to make in (2.35)-(2.36) replacements $2D \Rightarrow S$, $r_0 \Rightarrow q_0$, where q_0 is characteristic microscopic scale of charge transfer, while τ_0 has the same meaning as before. Thus, at $(\omega\tau_0)^2 \ll 1$ =,, we find

$$S_{\delta S}(\omega) \,=\, \frac{q_0^2}{3\tau_0 S}\cdot\frac{2\pi}{|\omega|}\,(\,\ln|\omega|\tau_0\,)^{-2} \qquad (2.38)$$

If the conductor represents a homogeneous sample with length L, then it is easy to transform (2.38) back to (2.35)-(2.36) by setting $q_0 = er_0/L$, where e is electron charge, and expressing S via diffusion coefficient and number of (independent) carriers. Here, q_0 is charge transported through leads in outer circuit (or short-circuit) when an inner carrier moves to distance r_0.

Next, consider, as an example, $p-n-$junction In this case $S = e^2 n$, where n is mean number of carriers crossing the junction per unit time. If carriers are not correlated one with another, then, evidently, we can write $q_0 \approx e$. Correspondingly, at $\tau_0 \sim 10^{-7}\,$s and $\omega/2\pi = 1\,$Hz we obtain from (2.38) estimate

$$S_{\delta S}(\omega) \,\approx\, \frac{0.002}{\tau_0 n}\cdot\frac{2\pi}{|\omega|} \qquad (2.39)$$

The scale τ_0 must be determined by "life-time" of carrier on the junction, so that $\tau_0 n \gg 1$. Relation of τ_0 to physical characteristic times of the system can be revealed in a more detail model only.

Of course, in the framework of the presented phenomenological statistical theory we can not generally expect estimates better than by an order of magnitude [7].

3. Current 1/f-noise in steady non-equilibrium state and nonlinear fluctuation-dissipation relations (FDR)

3.1. **Characteristic function of transported charge. Cubic FDR.** The use of mathematical formalism of non-Gaussian random processes gave us possibility to consider 1/f-noise in spite of leaving it in equilibrium state. Under switching on an electric field, fluctuations of diffusivities of carriers, $D(t)$, transform into fluctuations of their mobilities, $\mu(t)$, and thus of conductance, $g(t)$, and become source of current 1/f-noise. For correct analysis of this non-equilibrium state we shall apply the nonlinear fluctuation-dissipation relations (FDR) described in [16]-[21] ✓[73].

Let us return to above mentioned tw-terminal conductor. Let before time moment $t=0$ it is in equilibrium with its surroundings (and short-circuited) but after $t=0$ is subject to a given constant voltage $x(t)=\text{const}$. In such the case, as shown in [16,17] (see also [18,21]), due to the time reversibility of microscopic dynamics, the following exact generating FDR takes place [8]:

$$\left\langle \exp\left\{ \int_0^t \left[iu(t') - \frac{x}{T} \right] J(t')\,dt' \right\} \right\rangle_x = \\ = \left\langle \exp\left\{ - \int_0^t iu(t-t')\,J(t')\,dt' \right\} \right\rangle_x , \tag{3.1}$$

where $u(t)$ is arbitrary probe function. The subscript "x" reminds of non-equilibrium character of fluctuations. Taking here $u(t)=u=\text{const}$, introduce CF

$$\Delta_t(iu\,|\,x) \equiv \frac{1}{t} \ln \langle \exp\{ iu \int_0^t J(t')\,dt' \} \rangle_x \tag{3.2}$$

In equilibrium case it coincides with (2.5). From (3.1) it follows that

$$\Delta_t(iu - x/T\,|\,x) = \Delta_t(-iu\,|\,x) \tag{3.3}$$

Expansion of (3.3) into series over iu implies an infinite chain of FDR connecting average value, $\langle Q(t)\rangle_x$, of charge transported during observation time t, and various

[7] The authors of [1] quite successfully describe 1/f-noise in non-uniform semiconductor structures on the base of empirical model of mobility fluctuations (see Introduction) with $a \approx 0.001$. Interesting exception is presented by diodes with the Shottky barrier [1].

[8] This formula concerns the conductor itself and all the surroundings in contact with it, assuming their mutual thermodynamical equilibrium at common temperature T before an external perturbation (electric voltage). It is asumed also that there is no external magnetic field.

cumulants of fluctuations $J(t)$ and $Q(t)$. With the help of these FDR, it can be shown, in particular, that

$$\langle\, Q(t)\,\rangle_x \,=\, \sum_{m=1}^{\infty} c_m \left(\frac{x}{T}\right)^{2m-1} \langle\, Q^{(2m)}(t)\,\rangle_x \;, \tag{3.4}$$

where

$$\langle\, Q^{(n)}(t)\,\rangle_x \,\equiv\, \int_0^t \langle\, J(t_1)\,,\,\dots\,,\,J(t_n)\,\rangle_x \; dt_1 \dots dt_n \tag{3.5}$$

are n-order cumulants of transported charge, and the numbers c_n are defined by formulae

$$\sum_{m=1}^{\infty} c_m\, z^{2m-1} \,=\, \tanh\frac{z}{2}\;, \quad c_1 = \frac{1}{2}\,,\; c_2 = -\frac{1}{24}\,,\; c_3 = \frac{1}{240}\,,\,\dots$$

Differentiation of (3.4) in respect to t leads to fluctuational representation of average current:

$$\langle\, J(t)\,\rangle_x \,=\, \frac{x}{T}\int_0^t \langle\, J(t)\,,\, J(t^\prime)\,\rangle_x \; dt^\prime \,- \tag{3.6}$$
$$-\,\frac{1}{6}\left(\frac{x}{T}\right)^3 \int_0^t \langle\, J(t), J(t_1), J(t_2), J(t_3)\,\rangle_x \; dt_1 dt_2 dt_3 \,+\,\dots$$

Consider weakly non-equilibrium state and expand quantities in first row of (3.6) into series over x:

$$\langle\, J(t)\,\rangle_x \,=\, g_1(t)\,x \,+\, g_3(t)\,x^3 \,+\,\dots\;, \tag{3.7}$$

$$\langle\, J(t), J(t^\prime)\,\rangle_x \,=\, \langle\, J(t), J(t^\prime)\,\rangle_0 \,+\, x^2\, R(t,t^\prime) \,+\,\dots \tag{3.8}$$

(thus, for simplicity, we assumed the conductor electrically symmetric). Extracting in (3.6) linear terms, in respect to x, we find

$$g_1(t) \,=\, \frac{1}{T}\int_0^t \langle\, J(t), J(t^\prime)\,\rangle_0 \; dt^\prime \;, \tag{3.9}$$

that is usual fluctuation-dissipation theorem (FDT). Extraction of cubic terms yields "cubic FDT":

$$g_3(t) \,=\, \frac{1}{T}\int_0^t R(t,t^\prime)\, dt^\prime \,- \tag{3.10}$$
$$-\,\frac{1}{6T^3}\int_0^t \langle\, J(t), J(t_1), J(t_2), J(t_3)\,\rangle_0 \; dt_1 dt_2 dt_3$$

Notice that similar quadratic and cubic FDR were in part investigated by Efremov [9] and Stratonovich [20].

3.2. **Weakly non-equilibrium state. Statistical expression for correlation function of conductance fluctuations.** Let us consider relation (3.10) at $t \gg \tau_\mu$, where τ_μ is correlation time of equilibrium current noise in short-circuited conductor. The function $g_3(t)$, representing cubic part of average current response, covers contributions from both "fast" electric processes (responsible for conductivity in itself) and comparatively slow thermal processes, including change of conductance because of Joule heat. Hence, generally $g_3(t)$, in contrast to $g_1(t)$, is not constant at $t \gg \tau_\mu$ but can contain slow long "tail". Moreover, this tail may be non-stationary.

The function $R(t,t')$, as one can see from (3.8), describes "excess" fluctuations of current in weakly non-equilibrium state, i.e. in quadratic regime in respect to voltage (and mean current). Hence, in particular, $R(t,t')$ contains information about flicker fluctuations of current. From (3.10) and (2.18) it follows that low-frequency (flicker) current fluctuations in the quadratic regime are determined, first, by equilibrium fluctuations of white noise power and, second, by slow or non-stationary thermal processes (caused by Joule heating). Notice that the second source, possibly, is related to the so-called "temperature" 1/f-noise what is sometimes observed in metals (see Introduction) ✓[74].

Here, we are interested in the first source of current 1/f-noise having thermodynamically equilibrium nature. Therefore we shall neglect non-equilibrium thermal processes, i.e. suppose that at $t > \tau_\mu$ the system tends to strictly stationary state. In such the case $g_3(t)$ turns to constant at $t > \tau_\mu$. Besides, at $t\,, t' \gg \tau_\mu$ the function $R(t,t')$ becomes a function of the time difference only: $R(t,t') \equiv R(t-t')$. Then we can exploit he phenomenological concept of conductance fluctuations and write (at $t\,, t' \gg \tau_\mu$ and, clearly, $|t-t'| \gg \tau_\mu$)

$$R(t,t') \,=\, R(t-t') \,=\, K_g(t-t')\;, \tag{3.11}$$

where $K_g(t-t')$ is correlation function of these fluctuations[9]. Next, differentiate (3.10) in respect to t. Since at that the left side of (3.10) vanishes ✓[76], from (3.10)-(3.11) and (2.13) we obtain

$$K_g(t) \,=\, \frac{1}{2T^2}\int_0^t\!\!\int_0^t \langle\, J(t), J(t'), J(t''), J(0)\,\rangle_0 \; dt'dt''\;, \tag{3.12}$$

$$K_g(t) \,=\, K_S(t)/4T^2 \tag{3.13}$$

[9] Notice that use of model of conductance fluctuations automatically presumes stationarity. In other words, contribution of non-stationary processes (if any) to current 1/f-noise can not be described in this model surely exploited in the literature. ✓[75]

Thus, correlation function of conductance fluctuations (caused by equilibrium processes) is expressed through fourth-order cumulant of equilibrium current fluctuations. In essence, formula (3.12) must be treated as statistical definition of $K_g(t)$, since the fourth equilibrium cumulant of current always can be represented in rigorous terms of statistical mechanics.

Because of (3.12) and the Nyquist formula, $S = 2Tg$, spectra of relative flicker fluctuations $g(t)$ and $S(t)$ are identical and (in weakly non-equilibrium steady state) both coincide with spectrum of relative fluctuations of current. From (2.38) and (3.12), in the quadratic regime, we obtain

$$S_J(\omega) \,=\, \overline{J}^2 \cdot \frac{q_0^2}{3\tau_0 S} \cdot \frac{2\pi}{|\omega|}\,(\,\ln\,|\omega|\tau_0\,)^{-2}\ , \tag{3.14}$$

where $\overline{J} = gx$. In case of homogeneous conductor in the approximation of independent carriers formulae (2.35), (2.37) and (3.12) yield

$$S_J(\omega) \,=\, \overline{J}^2 \cdot \frac{2\pi\, a(\omega)}{N\,|\omega|}\ \ , \quad a(\omega) \,=\, \frac{r_0^2}{6D\tau_0}\,(\,\ln\,|\omega|\tau_0\,)^{-2} \tag{3.15}$$

As was noticed above, this result is in rather satisfactory quantitative agreement with general empirical formula (1.1). Since we based om “first principles” only, such result can be qualified as important evidence in favor of our approach.

3.3. **Nonlinear conductor. Current 1/f-noise in non-Ohmic regime.** Our analysis is valid for any (nonlinear) conductor, but in the Ohmic (quadratic in respect to current) regime. For description of non-Ohmic regime it is necessary to take into account non-Gaussianity of white noise in itself which, according to the FDR [17,21], is in close relationships with dissipative non-linearity. It is natural to suppose that character of these relationships is not affected by slow flicker fluctuations. In the corresponding statistical model all kinetic parameters of white noise fluctuate in coordination with each other. This can be mathematically formulated (for steady state at $t \gg \tau_\mu$) as follows (compare with (2.9)):

$$\Delta_t(iu|x) \,=\, \frac{1}{t}\,\ln\langle \exp\{\int_0^t \xi(t')\,S(iu|x)\,dt'\,\}\rangle'_x\ , \tag{3.16}$$

where $\int_0^t \xi(t')\,dt'$ is already considered process of scale-invariant diffusivity fluctuations, with $\xi(t) = S(t)/S$, and $S(iu|x)$ is CF of white noise ✓[77]. The latter, according to (3.3), should satisfy similar relation

$$S(iu\,-\,x/T\,|\,x) \,=\, S(-iu\,|\,x) \tag{3.17}$$

Of course, in general S, as well as r_0 and τ_0, is dependent on x.

Let us write the white noise CF as series

$$S(iu|x) \,=\, iu\,\overline{J}(x) \,+\, \frac{(iu)^2}{2}\,S(x) \,+\, \dots\;, \tag{3.18}$$

where $\overline{J}(x)$ is average current, and $S = S(x)$ spectral power of noise. Combining (3.16) with (3.18), it is easy to obtain, for current flicker fluctuations,

$$K_J(t) \,=\, \overline{J}^2(x)\,\frac{2q_0^2(x)}{3\tau_0 S(x)}\,\left(\ln\,\frac{t}{\tau_0}\right)^{-1} \tag{3.19}$$

and, correspondingly,

$$S_J(\omega) \,=\, \frac{\overline{J}^2(x)\,q_0^2(x)}{3\tau_0 S(x)}\cdot\frac{2\pi}{|\omega|}\,(\,\ln\,|\omega|\tau_0\,)^{-2} \tag{3.20}$$

Thus, spectrum of current fluctuations is presented, as before, by expression like (2.38), but now with taking into account possible dependences on x. At that, FDR (3.17) dictates definite, may be rigid, relations between $\overline{J}(x)$ and $S(x)$.

For example, consider again an ideal semiconductor diode ($p-n-$junction) in non-Ohmic regime, or shot noise regime. In this case, as is known,

$$S(x) \,=\, e\,|\overline{J}(x)| \tag{3.21}$$

at $e|x| \gg T$. Assuming that $q_0 = e$ and τ_0 do not change, we find from (3.20)-(3.21) that

$$S_J(\omega) \,\approx\, \frac{e\,|\overline{J}(x)|}{\tau_0}\cdot\frac{2\pi\,a(\omega)}{|\omega|} \tag{3.22}$$

Hence, in nonlinear regime intensity of current flicker noise grows proportionally to mean current, that is merely to number of carriers crossing the junction per unit time (see [1]) ✓[78].

4. Conclusion

Let us resume main aspects and results of the present work.

1. The experimental situation around 1/f-noise does not fit in frames of traditional notions. The matter is in confusion between concepts concerning statistical properties of ensembles and concepts concerning individual charge carrier random walks. The confusion is unconsciously provoked by use of traditional Gaussian model of diffusion.

2. Our principal result is that 1/f-noise can be explained without attraction of special physical mechanisms, merely as generic property of random Brownian motion of charge carriers, i.e. the well known process involved in most of electric phenomena.

3 . We constructed a simple novel model of Brownian motion, exploiting only the same physical premises as the standard model do, but refusing assumption about Gaussian character of statistics of real Brownian motion, since this is physically senseless idealization ✓[80]. Thus our theory is more adequate (though, of course, may require further improvements).

4 . The constructed theory inevitably leads to conclusion that real Brownian motion always possesses fluctuations of "rate of diffusion" with 1/f-type spectrum.

In the ensemble language one can say that diffusion coefficient and spectral power of "white" electric noise undergo fluctuations with 1/f-type spectrum. In non-equilibrium states these fluctuations manifest themselves in mobilities of carers, conductance and current.

5 . Random motion of separate carrier (and, generally, any concrete Brownian trajectory) has no certain diffusivity (diffusion coefficient). What is for the "rate of diffusion", its spontaneous (thermodynamic) fluctuations have no characteristic time scale, since make not back impact upon dynamical picture of motion or thermodynamical state of the system.

1/f-noise is just result of absence of long-living correlations in mechanisms of random motion of carriers (and has no relation to some macroscopically large relaxation times). Therefore, spectral contents of 1/f-noise is indifferent to system's sizes.

6 . Intensity of 1/f-noise is determined by microscopic scales only. In the constructed model, there are only two such parameters which indicate spatial and temporal lower bounds of scale invariance of Brownian motion.

7 . The resulting estimate of 1/f-noise intensity is in satisfactory agreement with experiments. At that, origin of the small empirical "Hooge constant" is revealed.

8 . A relevant microscopic information about 1/f-noise in stationary states can be obtained, in principle, from fourth-order equilibrium correlators (cumulants), with use of rigorous methods of statistical mechanics. This is conceptually quite new problem of theoretical physics.

Appendix

Let us show now in the spectral representation, i.e. in the frequency domain, that fluctuations of power of non-Gaussian equilibrium white noise and corresponding conductance fluctuations possess 1/f-type spectrum.

Let $J(\omega)$ be Fourier transform of current fluctuations. According to the Wiener-Khinchin theorem,

$$\langle\, J(\omega), J(\omega')\,\rangle_0 \;=\; 2\pi\, S\, \delta(\omega+\omega')\ ,$$

where $S = \mathrm{const}$ for white noise. From here it follows that

$$\begin{aligned}\langle\, J(\lambda^2\omega), J(\lambda^2\omega')\,\rangle_0 \;&=\; 2\pi\, S\, \delta(\lambda^2(\omega+\omega')) \;=\\ =\; 2\pi\, \frac{S}{\lambda^2}\, \delta(\omega+\omega') \;&=\; \left\langle \frac{J(\omega)}{\lambda}, \frac{J(\omega')}{\lambda} \right\rangle_0\end{aligned}$$

Hence,

$$\lambda\, J(\lambda^2\omega) \;\propto\; J(\omega)\ , \qquad (.1)$$

where symbol $\propto$ means identity in statistical sense. If current fluctuations have no coupling with some slow processes, then the scale invariance expressed by formula (.1) must extend to the whole statistics of the fluctuations. Taking this in mind, consider fourth-order spectral cumulant:

$$\langle\, J(\omega_1), J(\omega_2), J(\omega_3), J(\omega_4)\,\rangle_0 \;=\; 2\pi\, S_4(\omega_1,\omega_2,\omega_3)\, \delta(\omega_1+\omega_2+\omega_3+\omega_4)\ , \qquad (.2)$$

where S_4 is tri-spectrum. According to the analogue of (2.11) for current,

$$S_4(\omega_1,\omega_2,\omega_3) \;=\; S_S(\omega_1+\omega_2) \;+\; S_S(\omega_1+\omega_3) \;+\; S_S(\omega_2+\omega_3) \qquad (.3)$$

Replacing in (.2) $J(\omega)$ by $\lambda\, J(\lambda^2\omega)$ and combining the result with identity (.1) and expression (.2) itself, it is easy to deduce that

$$\begin{aligned}\lambda^2\, S_4(\lambda^2\omega_1, \lambda^2\omega_2, \lambda^2\omega_3) \;&=\; S_4(\omega_1,\omega_2,\omega_3)\ ,\\ \lambda^2\, S_S(\lambda^2\omega) \;&=\; S_S(\omega)\end{aligned}$$

From here we find

$$S_S(\omega) \;=\; \frac{\mathtt{const}}{|\omega|}\ , \qquad (.4)$$

that is the power and conductance fluctuations have 1/f-type low-frequency spectrum (solution $S_S(\omega) \propto \delta(\omega)$ is not appropriate since S_4 must be a smooth function). Our above results differ from this more formal one only by slight distortion of ideal scale invariance.

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[49] To be more precise, in electrolytes a may achieve as large values as $a \sim 10$ [24].

Useful information about temperature dependence of a in semiconductors can be found in [50].

[50] F. N. Hooge, IEEE Trans. El. Dev. **41**, No.11, 1926 (1994).

[51] Unceasing search of "fluctuation modes", or "fluctuators", to form 1/-type spectru, continues up to now. But, strangely, its otioseness does not push to simple idea that "fluctuators" are merely unique trajectories ("fates") of charge carriers.

[52] Highly dubious hypothesis ignoring the fact that energy (number of phonons) in any partiular vibration (phonon) mode possesses rather short (at least, not giant) relaxation, or thermalization, time (while fluctuations in summary energy density of different phonon modes are the same as the temperature fluctuations already shown to fail as a candidate for source of 1/f-noise).

In reality, as shown in [29], observed 1/f-noise in (equilibrium) phonon system represents fluctuations of rates of relaxation (relaxation times) of phonon modes (see also discussion in [42]).

[53] From viewpoint of the standard naive logics (see e.g. [54]), this paradox enforces to search slow "fluctuators" what randomly modulate conductance with magnitude inversely proportional to $\sqrt{N}$, as if number of fluctuators was proportional to N. Rather difficult task (ensuring employment of the logicians for many years in future)!

More adequate logics should take into account that slowly decaying statistical correlations in general do not presume slowly decaying cause-and-consequence correlations (long-living memory). In opposite, long statistical correlations (or even not decaying ones) can arise from indifference of system to pre-history of transport processes in it (i.e. just from absence of long memory)! For explanations see e.g. [28, 37] and introductory and/or discussion-conclusion sections in other our works, [27] - [**?**].

In brief, if charge conduction is indifferent to amount of already transortted charge $Q(t)$, then it is unable to ensure well certain value of time-averaged current $Q(t)/t$, which just means presence of arbitrary long statistical correlations between time-local current values.

That is what must be expected from rigorous statistical mechanics [26], in place of its crucial roughenings in conventional kinetics (pathological simplifications "mutilating" the very way of thinking of theoreticians).

[54] M. B. Weissman, Rev. Mod. Phys., **60**, 537 (1988).

[55] One can not break long correlations in a physical process which by itself is broken into many "short" random events! Long statistical correlations between constituent events of such process say only that all the events are equally responsible for deviations of the whole proces from some "average" regime. The deviations, in turn, result from independence of presently occuring events on total number and time distribution of previously happaned ons.

Such "purely random" process has no certain *a priori* "probabilities" for constituent events (free flights and collisions of carriers, or their passes through a sample, etc.). Therefore an adequate theory should start from dynamical (Hamiltonian) model or, at least, deal with full random realizations of the process as the whole, while use of a stochastic model with *a priori* postulated partial probabilities automatically "kills" 1/f-noise in theory and makes it a mystery in practice.

[56] Quantities like D and S, i.e. spectral densities, by their nature are time-nonlocal: they are attributes of not definite time moments but time intervals essentially greater than characteristic memory life-time of system under consideration. Therefore, they are out of scope (control) of the system. Consequently, their fluctuations do not cause system's back reaction and hence are scaleless (indifferent to time averaging).

Similarly, if you constantly forget your past, then you can not plan your future, and uncertainty of results of your activity will be indifferent to its duration.

Generally, any real (physical) memoryless random process possesses long-living scaleless ("flicker") fluctuations in its kinetic properties and thus is **non-ergodic** (terrible word!) in respect to them.

[57] Here, the statistical independence is taken in mind, i.e. independence in the sense of probability theory where it **by its definition** means that joint probability distribution of different increments is factored into product of their particular (marginal) distributions. In physics, scientists naturally take in mind "independence" in physical cause-and-consequence sense. The great mistake in practical statistical physics of noise, fluctuations and transport phenomena is common opinion that the "physical independence" can be identified with statistical one, thus giving rights to replace "honest" analysis of many-particle dynamics by 'hand-made" probability-theoretical models composed of *a priori* prescribed probabilities and glued by statistical independency. However, Krylov showed that this is wrong opinion, and rigorous statistical mechanics does not give such rights [26]. There phase trajectories of many-particle system can not be divided into statistically independent "elementary" events. Or, in other words, real events have no definite *a priori* probabilities, while their *a posteriori* probabilities are different at different phase trajectories (experiments).

In the commented preprint, the $r(t)$'s increments will loss their statistical independence and become all mutually correlated immediately when the kernel $G(r)$ from (2.17) will acwuire time dependence in (2.24).

[58] To be more precise, in general there are more than one characteristic "free path" lengths, and apparently r_0 can not be smaller than minimum of them. For example (touched in Introduction and below), if energy relaxation of carriers is more slow than velocity relaxation, then we have at least two different free paths, "elastic" and "inelastic". Then it seems natural to equate r_0 to shortest of them ("elastic"), while associating $\sqrt{2D\tau_0}$ with the longest ("inelastic"), so that $r_0^2/2D\tau_0 = D'/D \ll 1$, i.e. diffusivity 1/f-noise is much weaker than its "seed" refernce level corresponding to $D'/D = 1$.

The opposite situations, when $r_0^2/2D\tau_0 = D'/D \gg 1$, are possible too. Some examples can be found in [24, 25, 41]. Besides, recollect that in metals notably below Debye temperature, vice versa, velocity (direction) relaxation is slower than energy relaxation.

[59] This CF represents peculiar case of stable distributions, or the "α-stable Levy distributions", which corresponds to the "stability parameter" value $\alpha = 2 - 0$, thus suggesting alternative to the Gaussian distribution for which $\alpha = 2$.

Interestingly, this "quasi-Gaussian" [25, 28] distribution for the first time had appeared just in the commented preprint and independently, in same 1982, in mathematical work [60] (see also remarks in [28]). In the standard classification of stable distributions (see e.g. [12]) such one is absent. In our considerations, it plays role of intermediate idealization between absolutely nonphysical Gaussian distribution and physically meaningful distribution characterized by (2.33) or, more generally, (2.24)-(2.28).

Qualitative difference between random walk processes represented by CF (2.23) and CF (2.33) (or (2.24)-(2.28)) is that increments of the first of them are purely statistically independent and have infinite variance, while increments of the second have finite variance and all are essentially statistically dependent, regardless of their time separation. Thus, our cut-off of scale invariance ("fractality" [25]) of random walk $r(t)$ from below (at small scales) completely changes its statistical contents, especially in physical applications.

Unfortunately, such treatment of "fractal random walks" is out of traditions of "mathematical physics" which develops in the firm belief that dynmical random walks in statistical mechanics always can be decomposed into "independent" fragments and thus reduced to stochastic random walks like e.g. "Levy flights" (perhaps, just by this reason the "Journal of Statistical Physics" resembles sooner a "book of problems" from probability theory than researches statistical mechanics). However, as was predicted by Krylov [26], that are vain hopes, first of all in respect to "molecular random walks" in many-particle systems [31, 32, 34]. Applying the A. Einstein's sentence, *"God does not play dice"*!

[60] R.D.Hughes, E.W.Montroll and M.F.Shlesinger, J.Stat.Phys. **28** 111 (1982).

[61] This is key statement and key result of the preprint under consideration. Importantly, it appeals to rigorous statistical mechanics of many-particle systems, instead of probability-theoretical speculations. Now, we can state that statistical mechanics indeed justifies this appeal (see [27] - [39], [40, 42, 44] and references therein).

[62] As was shown in [25], probability weight of non-Gaussian (more or less long) tails of the $r(t)$'s distribution typically is around 0.03. Nevertheless, this means that probabilities of large deviations of $r(t)$ from typical diffusion rate (i.e. probabilities of $|r(t)| \gg \sqrt{2Dt}$) always are very mich greater than in Gaussian model (see [31, 37, 39, 48]).

Of course, statistics of large deviations is sensitive to details of the function $F(z)$, especially if it decays by a power law, e.g. $F(z) = 1/(1+z/\beta)^{1+\beta}$ (we took into account conditions (2.27)). At that, however, to satisfy actual finiteness of all $r(t)$'s cumulants, we should multiply such function of $z = r^2/4D't$ b a cut-off factor $F'(r^2/v_0^2t^2)$, where $F'(z')$ is fast decreasing function of $z'r^2/v_0^2t^2$ and v_0 is characteristic velocity of carriers (e.g. thermal or Fermi velocity). Then probability distribution of $r(t)$ asymptotically (at $v_0^2t^2/D't \to \infty$) obeys scale invariance $r^2 \propto t$ but far tails of this distribution and its higher-order cimilant significantly deviate from $r^2 \propto t$ (therefore, in particular, the "noise of 1/f-noise" [25] can be of $1/f^\gamma$-type with $\gamma > 1$). A similar picture arises in statistical-mechanical theory of "molecular random walks" [30, 31, 32, 34, 39, 48].

[63] Of course, that are not absolutely convincing reasonings, and one can suggest contrary instances to them. For example, in metals, due to the Fermi statistics of carriers, even under simple inelastic scattering (above the Debye temperature) $r_0^2 \approx \lambda_0^2 \gg 2D\tau_0 \approx 2D\tau_\mu$, so that the Hooge constant rises nearly T_F/T times as compared to the "seed" level [24] (as if only $\sim (T/T_F)\,N$ of total number N of conduction electrons, - from vicinity of Fermi surface, - effectively contribute to 1/f-noise).

On some of other possible variants of relationships between characteristics micro-scales see [24, 25, 28, 30, 31, 40, 41]. Unfortunately, exact equations of microscopic Hamiltoniam model of electron's random walk in phonon field [42] still are not analysed up to quantitative estimates to compare with the commented phenomenology.

[64] Though, strictly speaking, this integrability is not consequence of stationarity in its physical sense. In other words, 1/f-noise can look formally non-stationary even in thermodynamically equilibrium (all the more, in steady non-equilibrium) systems! This important theoretical fact was demonstrated and explained already in [27] (se also [28, 30]).

[65] On this subject see also [27, 28].

[66] This seeming paradox is main (practically fatal!) obstacle in ways to adequate theoretical perception of 1/f-noise.

Though, it is trivial logical truth that (long) statistical correlations can take place without (long) physical (cause-and-consequence) correlations beyond them. Moreover, forgetting of the past (history of a transport process) means impossibility to keep the future (fluctuations in transport rate) predictable and controllable. Therefore short-memory transport process inevitably acquires scaleless long-term fluctuations (and thus statistical correlations) of its instant rate.

Nevertheless, physicists (even theoreticians) accept for a fact the prejudice that physical independence is the same as statistical independence. Consequently, they continue, like thirty years ago, to search sources of 1/f-noise observed in transport processes (and other irreversible and transition processes) everywhere except randomness of these processes in itself!

Notice that measurements of effective "Hooge constant" ("Hooge parameter") in very different systems, - including modern small-size diamond, grathene and carbon nanotube based devices (see, for example, [67], [68] and [69, 70], respectively, and references in [41]), - usually show values from rather narrow interval $10^{-4} \lesssim a \lesssim 10^{-1}$, in spite of giant variety of conditions and regimes of charge transport under investigations. But, strikingly, this fact does not enforces investigators to assume that mysterious source of 1/f-noise is neither charge traps nor structural defects or impurities (or even phonon energy fluctuations, - see discussions in [29, 42]). That is what the Koz'ma Pritkov's joke is about (see epigraph above).

In reality, of course, the enumerated factors, along with many others, - e.g. Coulomb interaction between carriers and related screening and space charge effects, - all can essentially influence

1/f-noise, but this does not mean that they are its sources (similarly, they can influence white noise although not being its sources).

Hence, the actual problem waiting for theoreticians is development of dynamical (i.e. statistical-mechanical) models of systems under practical interest (thus continuing our first steps made in [24], [27] - [42], [48]).

The phenomenology we are commenting now is too primitive from viewpoint of such complicated inhomogeneous structures as mentioned various field-effect transistors [67, 68, 69, 70]. Nevertheless, it can be notably helpful in interpretation of experimental data (see comment [78] below).

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[69] J. Ma1nnik, I. Heller, A.M. Janssens, S.G. Lemay, and C. Dekker, Nano Lett. **8** (2) 685 (2008).

[70] Ju Hee Back, Sunkook Kim, Saeed Mohammadi, and Moonsub Shim, Nano Lett. **8** (4) 1090 (2008).

[71] “Rate of diffusion” by its sense is a time-nonlocal quantity smeared over time intervals much longer than system’s memory for diffusion. That is why this quantity has no definite instant values, is free of system’s back reaction, and fluctuates with no certain time scale.

[72] On this subject see also [29] and discussions in [28, 42].

[73] In English, see also papers [45] plus [46] - [48] and other our works on the nonlinear FDR, or “generalized FDR”, referenced in [46] - [48].

[74] See additionally [24] and references therein. Some usefull information can be found also in [54].

[75] Though, see above comment [64]. Here, we again have to distinguish between formal and physical stationarities, since they are not equivalent, and integral in (3.12) can be unboundedly growing function of t although being characteristics of equilibrium state. This means [27] that result of conductance measurement depends on its duration t but in no way on time when it begins, and at that does not improve when t increases (in opposite, longer time averaging makes it worse, since increases its mean-square relative deviation from ensemble average!). In such case (3.12) must be treated like the structure function of non-stationary random process (for examples see [27, 30]), and relations between theoretical expressions and experimental procedures need in more careful analysis.

[76] To be precise, it anyway decreases with time faster than $\propto 1/t$, while $K_g(t)$ by its sense decreases slower than $\propto 1/t$ (if not increases, - see above).

[77] Clearly, the CF $S(iu|x)$ here is the same as $\Delta_\infty(iu|x)$ in absence of low-frequency flicker fluctuations which are separated into factor $\xi(t)$. At that, the latter can be treated as derivative of randomly non-uniform “inner time” of the transport process, $\int_0^t \xi(t')\,dt'$, in respect to real time.

Similar separation of high-frequency noise and 1/f-type noise appears also, in the course of naturalal approximations, in the microscopic statistical-mechanical theory of molecular Brownian motion (see e.g. [30, 31, 39, 48]).

At the same time, importantly, formula (3.20) can be derived without such separation, by using the equilibrium scale invariance reasonings plus some FDR only (see [24]).

[78] Though, after introducing effective number of carriers under simultaneous transitions through the junction, $N(x) = |\overline{J}(x)|\tau_0/e$, and rewriting $e|\overline{J}(x)|/\tau_0 = \overline{J}^2(x)/N(x)$, formula (3.22) looks like in Ohmic regime.

The general formula (3.20) covers cases when, firstly, the conductor is neither discrete junction nor homogeneous distributed sample and, secondly, charge carriers can interact one with another, that is be (physically and statistically) mutually dependent.

In particular, we may apply formula (3.20) to field-effect transistors (FET), as far as it is possible to neglect their gate currents and treat FET like two-terminal conductor with variable properties tuned by gate voltage V_g. At that, according to experimental observations (see e.g. [67] - [70]), usually $S_J \propto \overline{J}^2$ at fixed V_g. Therefore, firstly, (3.20) can be written as

$$A(x) \equiv \frac{f\, S_J}{\overline{J}^2} = \frac{q_0^2(x)\, a_0}{S(x)\, \tau_0(x)} ,$$

where now $x = V_g - V_{th}$ is operating gate voltage counted from some "threshold voltage" V_{th}, $f = \omega/2\pi$, $a_0 = (1/3)(\ln|\omega|\tau_0)^{-2} \sim 0.001$ is the seed Hooge parameter, and $S(x)$ as before is white noise power spectral density (of source-drain current). Secondly, we can use the Nyquist formula, $S(x) = 2TG(x)$, with $G(x)$ being source-drain conductance.

For instance, consider charge transport in single-wall carbon nanotube (SWCNT) based FET. If length L of SWCNT and x are large enough, then $G \approx G_0 \equiv (4e^2/2\pi\hbar)\,\lambda/L$ [79], where λ, as above in Sec.2.7, is mean free path. As there, we have to take $q_0 = e\, r_0/L$ and naturally assume that $r_0 = \lambda$ and $\tau_0 = \lambda/v_F$, where v_F is Fermi velocity. Then

$$A = \frac{\pi\hbar\, v_F}{4TL} \cdot a_0 = \frac{a}{N} = \frac{e^2}{2TC_q} \cdot a_0 ,$$

where $N = (2/\pi\hbar v_F)\, e|x|L$ is number of carriers (with $2/\pi\hbar\, v_F$ representing 1D density of states in "hyperbolic" or "conical" band), $a = (e|x|/2T)\, a_0$ (with $e|x|$ representing Fermi energy, so that ratio a/a_0 looks like for metals), and $C_q = e^2\,(2/\pi\hbar\, v_F)\, L$ is SWCNT's "quantum capacitance".

For numerical estimates, we must take into account that $v_F \sim 10^8$ cm/s [79], and, - in view of just made assumptions, - consider comparatively long tubes with $L \sim 3\,\mu$m [69, 70] at sufficiently large $|x| \gtrsim 0.3\,$V (see e.g. Fig.1 in [70] for visuality). Then the above formula yields $A \sim 10^{-5}$. It agrees with A measured in [69], as well as with minimum (measured under $|x| \gtrsim 0.3\,$V) of A values found in [70]. At that, $N \sim 2000$ and $a \approx 10\, a_0 \sim 0.01$, in good agrreement with values $a \sim 0.01$ typically observed in [70].

At $|x| \lesssim 0.3\,$V, in [70] significantly greater A and a values were obtained. Apparently, they have subjective reason: the number of carriers there was estimated like we have done above, by $N = C_q|x|/e \equiv N_0(x)$, although in reality at relatively small $|x|$ (when metallic 'hyperbolic-band" conductance through SWCNT channel still is not formed) $N = N(x)$ decreases in a more fast way. Namely, such that $N(x)/N_0(x) \sim G(x)/G_0 \ll 1$ at small $|x|$, as simple analytical consideration does show (at that $G(x) \propto \overline{J}$ changes like current at Fig.1 in [70], and $A^{-1}(x) \propto G(x)$). By this reason, a appears to be strongly overestimated. Besides, objectively, high a values may be attributed to inhpmogeneity of gating the channel at small $|x|$ (since any inhpmogeneity lowers conductance and rises effective Hooge parameter).

Analogous satisfactory and even better situations, - as to agreement of our theory with experiment, - take place in cases of FET exploiting "H-terminated diamond" and graphene micrometer-size sheets [67, 68]. All that will be considered elsewhere.

[79] M.J. Biercuk, Sh. Ilani, Ch.M. Marcus, and P.L. McEuen, in [A.Jorio, G.Dresselhaus, M.S.Dresselhaus (Eds.). Carbon Nanotubes. Topics Appl. Physics **111**, 455493 (2008), Springer-Verlag, 2008].

[80] It is tercentennial heritage of the "law of large numbers" mathematically deduced by J. Bernoully [81] from assumption that seemingly (physically) independent observations (events) are statistically independent (in the sense of modern probability theory). This assumption allowed Bernoulli to manage with a small set of particular *a priori* "probabilities of events" and to propose their determination by means of time averaging. Indded, otherwise it would be necessary to introduce probabilities of pairs, ternaries, etc. (up to infinity), of events, so that number of ndeterminates always would stay greater than number of observations.

However, in the twentieth century N. Krylov [26] showed that in statistical mechanics such assumptin in general is wrong, that is "probabilities of events" (relative frequencies of events) significantly vary from one phase trajectory (experiment) to another, regardless of duration of timr averaging. Thus, the "law of large numbers" does not work, and statistics of real "flows of events" is essentially non-Gaussian even in low-frequency domain (see also remarks in introduction sections of [30, 34, 35]).

Therefore, A. Kolmogorov [82] was right recommending scientists be careful with assumptions about "independency" of physical random events.

At this point, it is time to follow the Koz'ma Prutkov's advice: *"If you have a fountain, stop it up. Let it too have a rest"*.

[81] Jacobi Bernoulli. Ars conjectandi. Basel, 1713 [in English: Jacob Bernoulli and E.D.Sylla (translator). Art of conjecturing. John Hopkins Univ., 2005; in Russian: Ya. Bernulli. O zakone bol'shih chisel. Moscow, Nauka, 1986].

[82] A.N. Kolmogorov. Foundations of the theory of probability. Chelsea, New York, 1956 (Russian original first was published in 1936).

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